A Generalization of the Banach Contraction Principle

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Abstract: In a complete metric space (X, d), we define contraction factor functions $\alpha : X \times X \to [0, \infty)$ and ω-distance functions $\rho : X \times X \to [0, \infty)$, of which the distance function d is a special case, such that if

$$\rho(Ax, Ay) \leqslant \alpha(x, y)\rho(x, y)$$

for all $x, y \in X$, then $A : X \to X$ has a (unique) fixed point.

Keywords: Banach Contraction Principle, ω-distance, fixed point.

1 Introduction

The Banach Contraction Principle states that if $A : X \to X$ is a mapping of a complete metric space (X, d) into itself, and there exists a number $0 \leqslant \alpha < 1$ such that for every two points $x, y \in X$:

$$d(Ax, Ay) \leqslant \alpha d(x, y), \tag{1}$$

then A has a unique fixed point, i.e., there exists a unique $x \in X$ satisfying $Ax = x$. A function $A : X \to X$ satisfying (1) is a contraction; the number $\alpha \in [0, 1)$ is the contraction factor.

Rakotch (1962) considers the problem of defining contraction factor functions $\alpha : X \times X \to [0, \infty)$ such that the Banach Contraction Principle remains valid when the constant α in (1) is replaced by a function $\alpha(x, y)$. This note considers also other functions than the distance function d in (1) under which the existence of a (unique) fixed point is guaranteed. More precisely, we define contraction factor functions $\alpha : X \times X \to [0, \infty)$, similar to those in Rakotch (1962), and functions $\rho : X \times X \to [0, \infty)$, of which the distance function d is a special case, in such a way that if

$$\rho(Ax, Ay) \leqslant \alpha(x, y)\rho(x, y)$$

for all $x, y \in X$, then $A : X \to X$ has a (unique) fixed point. The functions ρ are so-called ω-distances, introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi (1996), Suzuki and Takahashi (1996), and Suzuki (1997).

The set-up of the note is as follows. Section 2 recalls the definition of ω-distances and contains preliminary results. Section 3 presents the generalization of the Banach Contraction Principle. Section 4 contains concluding remarks.

2 Preliminaries

$\mathbb{N}$ denotes the set of positive integers. Let X be a metric space with metric d. Following Kada *et al.* (1996, p. 381), we define an ω-*distance* on X to be a function $\rho : X \times X \to [0, \infty)$ such that:

- ρ satisfies the triangle inequality, i.e., $\forall x, y, z \in X : \rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$;

- $\rho(x, \cdot) : M \to [0, \infty)$ is lower semicontinuous for every $x \in X$, i.e., if $y_m \to y$, then $\rho(x, y) \leqslant \liminf_{m\to\infty} \rho(x, y_m)$;

- for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x, y, z \in X$: if $\rho(z, x) \leqslant \delta$ and $\rho(z, y) \leqslant \delta$, then $d(x, y) \leqslant \varepsilon$.

The metric d is an ω-distance. Examples of many other ω-distances are found in Kada *et al.* (1996) and Suzuki and Takahashi (1996, Lemma 1). Kada *et al.* (1996, Lemma 1) prove:

Lemma 2.1 *Let (X, d) be a metric space and ρ an ω-distance on X. Consider points $x, y, z \in X$, a sequence (x_n) in X such that $x_n \to x$, sequences (α_n) and (β_n) in $[0, \infty)$ converging to zero. The following claims hold:*

(a) *If $\rho(x_n, x_m) \leqslant \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence in (X, d).*

(b) *If $\rho(x_n, y) \leqslant \alpha_n$ and $\rho(x_n, z) \leqslant \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $\rho(x, y) = \rho(x, z) = 0$, then $y = z$.*

Following Rakotch (1962), we define a family of functions that will take over the role of the contraction factors in the original statement of Banach's Contraction Principle.

Definition 2.2 Let (X, d) be a metric space and ρ an ω-distance on X. Denote by $F(\rho)$ the family of functions α on $X \times X$ satisfying the following conditions:

(a) for each $(x, y) \in X \times X$, $\alpha(x, y)$ depends only on the ω-distance $\rho(x, y)$; with a slight abuse of notation, this allows us to write $\alpha(\rho(x, y))$ instead of $\alpha(x, y)$;

(b) $0 \leqslant \alpha(d) < 1$ for every $d > 0$;

(c) $\alpha(d)$ is a decreasing function of d: if $d_1 \leqslant d_2$, then $\alpha(d_1) \geq \alpha(d_2)$.

3 A Contraction Principle

This section contains the statement and proof of our generalization of the Banach Contraction Principle.

Theorem 3.1 *Let (X,d) be a complete metric space, ρ an ω-distance on X, and $A : X \to X$ a function. If there exists an $\alpha \in F(\rho)$ such that*

$$\forall x, y \in X: \quad \rho(Ax, Ay) \leqslant \alpha(x,y)\rho(x,y), \tag{2}$$

then A has a unique fixed point x. This fixed point satisfies $\rho(x,x) = 0$.

Proof. Let $x_0 \in X$ and define for each $n \in \mathbb{N}$: $x_n = A^n x_0$. A simple inductive argument based on (2) and property (b) in Definition 2.2 yields that

$$\forall n \in \mathbb{N}: \quad \rho(x_{n+1}, x_n) \leqslant \rho(x_1, x_0), \tag{3}$$

and

$$\forall k, \ell, m \in \mathbb{N}: \quad \text{if } k > m, \text{ then } \rho(x_k, x_{k+\ell}) \leqslant \rho(x_m, x_{m+\ell}). \tag{4}$$

Let $\varepsilon > 0$. Define $R := \max\{\varepsilon, \frac{\rho(x_0,x_1)+\rho(x_1,x_0)}{1-\alpha(\varepsilon)}\}$. By property (b) in Definition 2.2, $\alpha(\varepsilon) < 1$, so R is well-defined. We claim that

$$\forall n \in \mathbb{N}: \quad \rho(x_0, x_n) \leqslant R. \tag{5}$$

Let $n \in \mathbb{N}$. Inequality (5) trivially holds if $\rho(x_0, x_n) \leqslant \varepsilon$, so assume that $\rho(x_0, x_n) > \varepsilon$. Consecutively using

- the triangle inequality for ρ,
- inequalities (2) and (3),
- the fact that $0 \leqslant \alpha(x_0, x_n) \leqslant \alpha(\varepsilon) < 1$, which follows from the assumption that $\rho(x_0, x_n) > \varepsilon$ and properties (b) and (c) in Definition 2.2,

the following chain of inequalities holds:

$$\begin{aligned} \rho(x_0, x_n) &\leqslant \rho(x_0, x_1) + \rho(x_1, x_{n+1}) + \rho(x_{n+1}, x_n) \\ &\leqslant \rho(x_0, x_1) + \alpha(x_0, x_n)\rho(x_0, x_n) + \rho(x_1, x_0) \\ &\leqslant \rho(x_0, x_1) + \alpha(\varepsilon)\rho(x_0, x_n) + \rho(x_1, x_0). \end{aligned}$$

Hence $\rho(x_0, x_n) \leqslant \frac{\rho(x_0,x_1)+\rho(x_1,x_0)}{1-\alpha(\varepsilon)} \leqslant R$, finishing the proof of (5). We proceed to prove that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall p \in \mathbb{N} : \quad \rho(x_N, x_{N+p}) < \varepsilon. \tag{6}$$

Let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that

$$R(\alpha(\varepsilon))^N < \varepsilon. \tag{7}$$

This is possible, since $0 \leqslant \alpha(\varepsilon) < 1$ by property (b) of Definition 2.2. Let $p \in \mathbb{N}$. For each $k = 0, \dots, N-1$, inequality (2) implies

$$\rho(x_{k+1}, x_{k+p+1}) \leqslant \alpha(x_k, x_{k+p})\rho(x_k, x_{k+p}).$$

Taking the product from $k = 0$ to $k = N-1$ and dividing both sides by the common term $\prod_{k=0}^{N-2} \rho(x_{k+1}, x_{k+p+1})$ yields

$$\rho(x_N, x_{N+p}) \leqslant \rho(x_0, x_p) \prod_{k=0}^{N-1} \alpha(x_k, x_{k+p}). \tag{8}$$

Division by the common term $\prod_{k=0}^{N-2} \rho(x_{k+1}, x_{k+p+1})$ is correct by definition if $\rho(x_{k+1}, x_{k+p+1}) > 0$ for every $k \in \{0, \dots, N-2\}$, but also if $\rho(x_{k+1}, x_{k+p+1}) = 0$ for a specific $k \in \{0, \dots, N-2\}$, inequality (8) remains valid. In this case, namely, inequality (4) yields that $0 \leqslant \rho(x_N, x_{N+p}) \leqslant \rho(x_{k+1}, x_{k+p+1}) = 0$, i.e., the left-hand side of (8) equals zero, whereas its right-hand side is always nonnegative. Combining (5) and (8):

$$\rho(x_N, x_{N+p}) \leqslant R \prod_{k=0}^{N-1} \alpha(x_k, x_{k+p}). \tag{9}$$

Discern two cases:

Case 1: If $\rho(x_k, x_{k+p}) < \varepsilon$ for some $k \in \{0, \dots, N-1\}$, then (4) yields that $\rho(x_N, x_{N+p}) \leqslant \rho(x_k, x_{k+p}) < \varepsilon$.

Case 2: If $\rho(x_k, x_{k+p}) \geq \varepsilon$ for every $k \in \{0, \dots, N-1\}$, then property (c) in Definition 2.2 implies that $\alpha(x_k, x_{k+p}) \leqslant \alpha(\varepsilon)$ for every $k \in \{0, \dots, N-1\}$. Using (7) and (9) yields

$$\rho(x_N, x_{N+p}) \leqslant R \prod_{k=0}^{N-1} \alpha(x_k, x_{k+p}) \leqslant R(\alpha(\varepsilon))^N < \varepsilon.$$

This proves (6). Statements (4) and (6) immediately imply

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \ \forall p \in \mathbb{N}: \quad \rho(x_n, x_{n+p}) < \varepsilon. \tag{10}$$

By (10), there exists a sequence (α_n) in $[0, \infty)$ converging to zero, such that

$$\forall n, p \in \mathbb{N}: \quad \rho(x_n, x_{n+p}) \leqslant \alpha_n. \tag{11}$$

But then (x_n) is a Cauchy sequence by part (a) of Lemma 2.1. Since (X, d) is complete, (x_n) has a limit $x \in X$. We show that $Ax = x$. Since $\rho(x_n, \cdot)$ is lower semicontinuous and $x_m \to x$, it follows from (11) that

$$\forall n \in \mathbb{N}: \quad \rho(x_n, x) \leqslant \liminf_{m \to \infty} \rho(x_n, x_m) \leqslant \alpha_n, \tag{12}$$

and, using (2) and (12), that

$$\forall n \in \mathbb{N}: \quad \rho(x_n, Ax) = \rho(Ax_{n-1}, Ax) \leqslant \rho(x_{n-1}, x) \leqslant \alpha_{n-1}. \tag{13}$$

From (12), (13), and part (b) of Lemma 2.1, it follows that $Ax = x$, i.e., that x is a fixed point of A. To see that $\rho(x, x) = 0$, suppose — to the contrary — that $\rho(x, x) > 0$. Then $0 \leqslant \alpha(x, x) < 1$ by property (b) of Definition 2.2; by (2) and the fact that x is a fixed point, it follows that:

$$\rho(x, x) = \rho(Ax, Ax) \leqslant \alpha(x, x)\rho(x, x) < \rho(x, x),$$

a contradiction. Finally, to prove that x in the *unique* fixed point of A, suppose that $y \in X$ satisfies $Ay = y$. Analogous to the proof that $\rho(x, x) = 0$, it follows that $\rho(x, y) = 0$, so part (b) of Lemma 2.1 implies that $x = y$. □

4 Concluding Remarks

Some remarks concerning the generalizations embodied in Theorem 3.1:

- If we take α to be a constant function in $[0, 1)$, we obtain Theorem 2 of Suzuki and Takahashi (1996);

- If we take $\rho = d$, we obtain the contraction theorem of Rakotch (1962, p. 463);

- If we take $\rho = d$ and take α to be a constant function in $[0, 1)$, we obtain the original Banach Contraction Principle.

If A itself does not satisfy (2), but some power A^n ($n \in \mathbb{N}$) of A does, the conclusion of Theorem 3.1 still holds: according to Theorem 3.1, A^n has a unique fixed point x, but

$$Ax = A(A^n x) = A^n(Ax)$$

indicates that Ax is also a fixed point of A^n. Hence $Ax = x$, i.e., x is a fixed point of A. The fact that x is the unique fixed point of A and $\rho(x, x) = 0$ follows in the same way as before.

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